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Energy Spectrum of the Relativistic Duffin-Kemmer-Petiau Equation

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Abstract The bound state energy eigenvalues for the relativistic DKP oscillator and DKP Coulomb potentials are determined by using an exact quantization rule. The corresponding eigenfunctions are also obtained. The results are consistent with those obtained by others methods.

Keywords Duffin-Kemmer-Petiau equation · Exact quantization rule · DKP oscillator · DKP Coulomb potential

1 Introduction

The Duffin-Kemmer-Petiau (DKP) [1–3] equation is a natural way to describe scalar and vector particles with the help of a covariant relativistic formalism. The DKP equation is the result of research, which followed the success of the Dirac equation of the spin 1/2 particle, in order to find a first-order equation of wave which describes the particles of spin 0 and spin 1. In other words, the DKP equation is a direct generalization of the Dirac equation to the particles of integer spin in which one replaces the gamma matrices by beta matrices but verifying a more complicated algebra known as DKP algebra. In addition, there has been a recent revival of interest in the DKP equation and its relevance to some problems in nuclear and particles physics.

One of the important tasks of relativistic quantum mechanics is to determine exact analytical solutions of the relativistic wave equation. Exact solutions of the relativistic Dirac, Klein-Gordon and Duffin-Kemmer-Petiau equations are possible only for certain potentials. Various methods have been used to obtain the exact bound state energy eigenvalues and corresponding eigenfunctions of the relativistic DKP equation.

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Recently, the Nikiforov-Ouvarov method has been employed to obtain the exact solution of the DKP equation in the presence of a deformed Hulthén potential [4], the asymptotic iteration method (AIM) is applied in [5] to solve the DKP oscillator and DKP Coulomb problems.

On the other hand, recently, Ma and Xu proposed an exact quantization rule proven without any approximations [6, 7], by which they determined the energy spectrum for one-dimensional finite square well potential, the harmonic oscillator, the Morse potential, the asymmetric Rosen-Morse potential, the first and second Pöschl-Teller potential. They generalized this quantization rule to the three-dimensional Schrödinger equation with spherically symmetric potential and they obtained the exact energy spectrum of the harmonic oscillator and the hydrogen atom. Using this method, Qiang *et al.* found arbitrary l-state solutions of the rotating Morse potential with the Pekeris approximation [8], and they also solved the energy spectrum problem for the relativistic rotational Morse potential with pseudospin symmetry [9].

We can also cite other works in this field [10, 11].

In this paper, we will use the Ma's exact quantization rule to solve the relativistic energy eigenvalue problem in the spin zero representation. First, we introduce the exact quantization rule and in the section 3, after a brief presentation of the DKP formalism, the second order differential equation for the DKP oscillator is derived and the energy eigenvalues are determined. In the following section we derive the second order radial differential equation for the DKP Coulomb potential and using the exact quantization rule we obtain the exact energy spectrum. The corresponding eigenfunctions are also determined for each potential.

2 The Exact Quantization Rule

We give a brief review of the exact quantization rule. Inserting the logarithmic derivative $\phi(x) = \frac{1}{\psi(x)}[\frac{d\psi(x)}{dx}]$ in the one dimensional Schrödinger equation ($\hbar = 2\mu = 1$)

$$\frac{d^2}{dx^2}\psi(x) = -[E - V(x)]\psi(x), \tag{1}$$

we obtain the Riccati equation

$$\frac{d}{dx}\phi(x) = -[E - V(x)] - \phi(x)^2, \tag{2}$$

where the potential $V(x)$ is a piecewise continuous real function of x. Furthermore, it is known from the Sturm-Liouville theorem that $\phi(x)$ decreases monotonically with respect to x between two turning points, where $E \geq V(x)$. Specifically, as x increases across a node of the wavefunction $\psi(x)$, where $E \geq V(x)$, $\phi(x)$ decreases to $-\infty$, jump to $+\infty$, and then decreases again. By carefully studying the one dimensional Schrödinger equation Ma and Xu proposed an exact quantization rule

$$\int_{x_L}^{x_R} k(x)dx = N\pi + \int_{x_L}^{x_R} \phi(x)\left[\frac{dk(x)}{dx}\right]\left[\frac{d\phi(x)}{dx}\right]^{-1} dx, \tag{3}$$

where $k(x) = \sqrt{E - V(x)}$ is the momentum, x_L and x_R are the classical turning points given by $E = V(x_L) = V(x_R)$, N is the number of nodes of $\phi(x)$ in the region $E \geq V(x)$ and is larger by one than the number of nodes of the wavefunction $\psi(x)$.

In the exact quantization rule (3), the first term $N\pi$ represents the contribution from the nodes of the wavefunction, and the second one is called the quantum correction. Ma and Xu have found that this quantum correction is independent of the node number of the wavefunction for the exactly solvable potentials. Thus, its can be replaced in equation (3) by

$$\int_{x_{L0}}^{x_{R0}} \phi_0(x)\left[\frac{dk_0(x)}{dx}\right]\left[\frac{d\phi_0(x)}{dx}\right]^{-1} dx, \tag{4}$$

where subscript 0 denotes the ground state [9].

For the three-dimensional Schrödinger equation with a spherically symmetric potential $V(r)$, after the separation of variables, the radial Schrödinger equation is writing

$$\frac{d^2}{dr^2}R(r) = -[E - U(r)]R(r), \quad U(r) = V(r) + \frac{l(l+1)}{r^2}. \tag{5}$$

Since (5) is similar to (1), the quantization rule (3) is generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential

$$\int_{r_L}^{r_R} k(r)dr = N\pi + \int_{r_{L0}}^{r_{R0}} \phi_0(r)\left[\frac{dk_0(r)}{dr}\right]\left[\frac{d\phi_0(r)}{dr}\right]^{-1} dr. \tag{6}$$

3 DKP Harmonic Oscillator

The first-order relativistic Duffin-Kemmer-Petiau equation for a free spin zero or spin one particle of mass m is

$$(c\beta.\mathbf{p}+mc^2)\psi = i\hbar\beta^0\frac{d\psi}{dt}, \tag{7}$$

where the matrices β^μ ($\mu = 0, 1, 2, 3$) verify the following commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu, \tag{8}$$

which defines the so-called Duffin-Kemmer-Petiau algebra. The algebra generated by the four β matrices has three irreducible representations. The physical ones are of dimensions 5 and 10 describing, respectively, particles of spin 0 and 1. For the case of spin 0, the explicit form of the β^μ is given by

$$\beta^0 = \begin{pmatrix} \theta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \beta^i = \begin{pmatrix} \mathbf{0} & \rho^i \\ -\rho_T^i & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3, \tag{9}$$

where the block elements are defined as

$$\begin{aligned} &\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\rho^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{10}$$

where ρ_T denotes the transposed matrix of ρ and $\mathbf{0}$ is the zero matrix. For the spin one particles, β^μ are 10×10 matrices (see [5]).

For the external potential introduced with the nominimal substitution [12]

$$\mathbf{p} \to \mathbf{p} - im\omega\eta^0\mathbf{r}, \tag{11}$$

where ω is the oscillator frequency and $\eta^0 = 2(\beta^0)^2 - 1$, the DKP equation is

$$(c\beta.(\mathbf{p} - im\omega\eta^0\mathbf{r}) + mc^2)\psi = i\hbar\beta^0\frac{d\psi}{dt}. \tag{12}$$

In the spin zero representation, the five component DKP spinor

$$\psi(\mathbf{r}) = \begin{pmatrix} \psi_{upper} \\ i\psi_{lower} \end{pmatrix} \quad \text{with } \psi_{upper} \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \text{ and } \psi_{lower} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \tag{13}$$

so that for stationary states the DKP equation can be written as [12]

$$mc^2\varphi_1 = E\varphi_2 + ic(\mathbf{p}+im\omega\mathbf{r}).\mathbf{A}, \tag{14}$$

$$mc^2\varphi_2 = E\varphi_1, \tag{15}$$

$$mc^2\mathbf{A} = ic(\mathbf{p}-im\omega\mathbf{r})\varphi_1, \tag{16}$$

where $\mathbf{A}$ represents the vector (A_1, A_2, A_3). Since the five-component wavefunction ψ is simultaneously an eigenfunction of J^2 and J_3,

$$J^2\begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} = \begin{pmatrix} L^2\psi_{upper} \\ (L+S)^2\psi_{lower} \end{pmatrix} = J(J+1)\begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix}, \tag{17}$$

$$J_3\begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix} = \begin{pmatrix} L_3\psi_{upper} \\ (L_3+s_3)\psi_{lower} \end{pmatrix} = M\begin{pmatrix} \psi_{upper} \\ \psi_{lower} \end{pmatrix}, \tag{18}$$

where the total angular momentum

$$J = L + S, \tag{19}$$

is a constant of the motion. For the spinless DKP oscillator, the general solution for a central problem is presented as follows [12]

$$\psi_{JM}(r) = \frac{1}{r}\begin{pmatrix} F_{nJ}(r)Y_{JM}(\Omega) \\ G_{nJ}(r)Y_{JM}(\Omega) \\ i\sum_L H_{nJL}(r)Y_{JL1}^M(\Omega) \end{pmatrix}, \tag{20}$$

where $Y_{JM}(\Omega)$ are the spherical harmonics of order J, $Y_{JL1}^M(\Omega)$ are the normalized vector spherical harmonics and F_{nJ}, G_{nJ}, H_{nJL} are radial wavefunctions. Inserting $\psi_{JM}(r)$ into (12), one obtains the following set of first-order coupled relativistic differential radial equations

$$EF = mc^2G, \tag{21}$$

$$\hbar c\left(\frac{d}{dr}-\frac{J+1}{r}+\frac{m\omega r}{\hbar}\right)F=-\frac{1}{\alpha_J}mc^2H_1, \tag{22}$$

$$\hbar c\left(\frac{d}{dr}-\frac{J}{r}+\frac{m\omega r}{\hbar}\right)F=\frac{1}{\zeta_J}mc^2H_{-1}, \tag{23}$$

$$-\alpha_J\left(\frac{d}{dr}+\frac{J+1}{r}-\frac{m\omega r}{\hbar}\right)H_1+\zeta_J\left(\frac{d}{dr}-\frac{J}{r}-\frac{m\omega r}{\hbar}\right)H_{-1}=\frac{1}{\hbar c}(mc^2F-EG), \tag{24}$$

with the definition

$$\alpha_J=\sqrt{(J+1)/(2J+1)},\qquad \zeta_J=\sqrt{J/(2J+1)}, \tag{25}$$

and

$$F_{nJ}(r)=F(r),\qquad G_{nJ}(r)=G(r),\qquad H_{nJJ\pm1}(r)=H_{\pm1}(r). \tag{26}$$

Inserting now (21)–(23) in (24), the homogeneous second-order differential equation for the DKP harmonic oscillator [12] is obtained as

$$\left(\frac{d^2}{dr^2}+\frac{E^2-m^2c^4}{(\hbar c)^2}+\frac{3m\omega}{\hbar}-\frac{m^2\omega^2r^2}{\hbar^2}-\frac{J(J+1)}{r^2}\right)F(r)=0. \tag{27}$$

Introducing

$$\varepsilon=(E^2-m^2c^4)(\hbar c)^{-2}+(3m\omega/\hbar)\quad\text{and}\quad \alpha=m\omega/\hbar, \tag{28}$$

and inserting this in (27) we have an equation

$$\left(\frac{d^2}{dr^2}+\varepsilon-\alpha^2r^2-\frac{J(J+1)}{r^2}\right)F(r)=0 \tag{29}$$

similar to the radial Schrödinger equation (5) and the associated Riccati equation is then

$$\frac{d}{dr}\phi(r)=-\left[\varepsilon-\alpha^2r^2-\frac{J(J+1)}{r^2}\right]-\phi(r)^2. \tag{30}$$

The insertion in the previous equation of a solution $\phi_0(r)$ of the form

$$\phi_0(r)=ar^{-1}+br,\quad a>0,\ b<0, \tag{31}$$

gives the algebraic equation

$$J(J+1)-a(a-1)+(\alpha^2-b^2)r^4-(b+2ab+\varepsilon_0)r^2=0. \tag{32}$$

This equation is verified for all the values of r when

$$a=J+1,\qquad b=-\alpha, \tag{33}$$

and when, the ground state energy is

$$\varepsilon_0=2\alpha(J+3/2). \tag{34}$$

Thus, the desired function is

$$\phi_0(r) = (J+1)r^{-1} - \alpha r. \tag{35}$$

Introducing a new variable of the form $y = r^2$, the integral of the momentum becomes

$$\int_{r_L}^{r_R} k(r)dr = \frac{\alpha}{2}\int_{y_L}^{y_R} \frac{dy}{y}\sqrt{(y-y_L)(y_R-y)}, \tag{36}$$

where the turning points $y_L = r_L^2$ and $y_R = r_R^2$ are given by the relations $y_L + y_R = \alpha^{-2}\varepsilon$ and $y_L y_R = \alpha^{-2}J(J+1)$. Using the following formula [14]

$$\int_{y_L}^{y_R} \frac{dy}{y}\sqrt{(y-y_L)(y_R-y)} = \frac{\pi}{2}(y_L+y_R) - \pi\sqrt{y_L y_R}, \tag{37}$$

we obtain for the first integral in the quantization rule

$$\int_{r_L}^{r_R} k(r)dr = \frac{\pi}{2}\left(\frac{\varepsilon}{2\alpha} - \sqrt{J(J+1)}\right). \tag{38}$$

With the help of the change of variable $y = r^2$, the second integral in the rule (6) becomes

$$\int_{r_{L0}}^{r_{R0}} \phi_0(r)\left[\frac{dk_0(r)}{dr}\right]\left[\frac{d\phi_0(r)}{dr}\right]^{-1} dr$$
$$= \int_{y_{L0}}^{y_{R0}} \frac{\alpha(J+1)dy}{f(y)} - \int_{y_{L0}}^{y_{R0}} \frac{J(J+1)dy}{2yf(y)} - \int_{y_{L0}}^{y_{R0}} \frac{\alpha^2 y dy}{2f(y)} - \int_{y_{L0}}^{y_{R0}} \frac{\alpha(J+1)dy}{(J+1+\alpha y)f(y)} \tag{39}$$

$$= \frac{\pi}{2}(J - \sqrt{J(J+1)} - 1/2), \tag{40}$$

where $f(y) = \sqrt{-\alpha^2 y^2 + \varepsilon_0\, y - J(J+1)}$.

The four integrals in the right hand side of (39) can be evaluated using the formulas given in [15]. Knowing that the nodes number of $\phi(r)$ is $N = n+1$ where $n = 0, 1, 2, \ldots$ denotes the radial quantum number, the exact quantization rule gives

$$\varepsilon_n = \alpha(2J + 4n + 3). \tag{41}$$

Substituting (28) in (41), we obtain the relativistic energy spectrum of the DKP oscillator

$$\frac{1}{2mc^2}(E_{NJ}^2 - m^2c^4) = N\hbar\omega, \tag{42}$$

where $N = 2n + J$ is the principal quantum number. The result is in agreement with those of [5] and [12].

In order to obtain the relativistic radial wavefunction for the scalar DKP oscillator, let us turn to the Schrödinger-like equation (29). According to [5], we take

$$F(r) = r^{J+1}e^{-\alpha r^2/2}f(r). \tag{43}$$

Substituting now (41) and (43) in (29), we obtain

$$\frac{d^2}{dr^2}f(r)+2\left(-\alpha r+\frac{J+1}{r}\right)\frac{d}{dr}f(r)+4n\alpha f(r)=0. \tag{44}$$

Defining a new variable $y=\alpha r^2$, the above equation becomes

$$y\frac{d^2}{dy^2}f(y)+\left(J+\frac{3}{2}-y\right)\frac{d}{dy}f(y)+nf(y)=0, \tag{45}$$

which is the well-known differential equation satisfied by the associated Laguerre polynomials, then we write

$$f(y)=L_n^{J+1/2}(y), \tag{46}$$

where $L_n^{J+1/2}$ is the associated Laguerre polynomials of order n, we may express the radial wavefunction as

$$F(r)=\lambda_{\text{norm}}\, r^{J+1}e^{-\alpha r^2/2}L_n^{J+1/2}(\alpha r^2), \tag{47}$$

where λ_{norm} is a normalization constant. The other radial wavefunctions $G(r)$ and $H_{\pm 1}(r)$ can be determined by using (21)–(23).

Our result is consistent with the result of Ref. [12].

4 DKP Coulomb Potential

We will apply in the present section the exact quantization rule for a spinless charged pion (π^-) in the presence of a Coulomb field. Using the following *ansatz* [5]

$$\begin{aligned} a_\pm&=\frac{mc^2\pm E}{\hbar c}, \qquad \gamma=\alpha Z, \qquad \lambda_\pi=\frac{\hbar}{mc},\\ \kappa&=\frac{2}{\hbar c}\sqrt{m^2c^4-E^2}, \qquad \xi=\frac{2\gamma E}{\kappa\hbar c}, \qquad \rho=\kappa r, \end{aligned} \tag{48}$$

the system of coupled equations for the Coulomb potential becomes

$$\alpha_J=\left(\frac{dF}{d\rho}-\frac{J+1}{\rho}F\right)=-\frac{1}{\kappa\lambda_\pi}H_1, \tag{49}$$

$$\zeta_J\left(\frac{dF}{d\rho}+\frac{J}{\rho}F\right)=\frac{1}{\kappa\lambda_\pi}H_{-1}, \tag{50}$$

$$-\alpha_J\left(\frac{dH_1}{d\rho}+\frac{J+1}{\rho}H_1\right)+\zeta_J\left(\frac{dH_{-1}}{d\rho}-\frac{J}{\rho}H_{-1}\right)=\kappa\lambda_\pi\left(\frac{a_+}{\kappa}+\frac{\gamma}{\rho}\right)\left(\frac{a_-}{\kappa}-\frac{\gamma}{\rho}\right)F. \tag{51}$$

Eliminating H_1 and H_{-1} in favor of F, we obtain the second-order differential equation [5]

$$\frac{d^2F(\rho)}{d\rho^2}+\left(-1/4+\frac{\xi}{\rho}-\frac{J(J+1)-\gamma^2}{\rho^2}\right)F(\rho)=0. \tag{52}$$

If we put $J(J+1)-\gamma^2=l(l+1)$ or $l=-\frac{1}{2}+\sqrt{(J+\frac{1}{2})^2-\gamma^2}$, (52) is written as follows

$$\frac{d^2F(\rho)}{d\rho^2}=-\left[-1/4+\frac{\xi}{\rho}-\frac{l(l+1)}{\rho^2}\right]F(\rho), \tag{53}$$

which is a particular case of the following equation

$$\frac{d^2F(\rho)}{d\rho^2}=-\left[\varepsilon_n+\frac{\xi}{\rho}-\frac{l(l+1)}{\rho^2}\right]F(\rho), \tag{54}$$

which is a Schrödinger-like equation.

Thus we can apply the exact quantization rule to the potential $V(\rho)=-\frac{\xi}{\rho}+\frac{l(l+1)}{\rho^2}$ given in equation (54), to obtain the equation satisfied by the parameter ε_n, nevertheless, at the end of calculus we must replace ε_n by the value $-1/4$.

For this purpose we first solve the Riccati equation

$$\frac{d}{d\rho}\phi(\rho)=-\left[\varepsilon_n+\frac{\xi}{\rho}-\frac{l(l+1)}{\rho^2}\right]-\phi(\rho)^2, \tag{55}$$

where $\phi(\rho)$ is the logarithmic derivative of $F(\rho)$. Inserting the solution with one node $\phi_0(\rho)=a\rho^{-1}+b$ with $(a>0)$ into the Riccati equation, we obtain $a=l+1$, $b=-\xi(2(l+1))^{-1}$ and the ground state energy $\varepsilon_0=-\xi^2(2(l+1))^{-2}$, thus the desired function is

$$\phi_0(\rho)=(l+1)\rho^{-1}-\xi(2(l+1))^{-1}. \tag{56}$$

Now, let us turn to the exact quantization rule (6). After an integration by parts, (6) can be rewritten as

$$\int_{r_L}^{r_R}k(r)dr=N\pi-\int_{r_{L0}}^{r_{R0}}k_0(r)dr+\int_{r_{L0}}^{r_{R0}}k_0(r)\frac{\phi_0(r)\phi_0''(r)}{\phi_0'(r)^2}dr, \tag{57}$$

where r_{L0} and r_{R0} are the turning points satisfying $k_0(r_{L0})=k_0(r_{R0})=0$ and the prime denotes the derivative with respect to r. Equation (57) is another form of the exact quantization rule, we will apply it in the present case by replacing the variable r in ρ. Substituting now the following expression

$$\frac{\phi_0(\rho)\phi_0''(\rho)}{\phi_0'(\rho)^2}=-\xi(l+1)^{-2}\rho+2, \tag{58}$$

in the equation (57), one obtains

$$\int_{\rho_L}^{\rho_R}k(\rho)d\rho=\int_{\rho_{L0}}^{\rho_{R0}}k_0(\rho)d\rho+(-\xi)(l+1)^{-2}\int_{\rho_{L0}}^{\rho_{R0}}\rho k_0(\rho)d\rho+(n+1)\pi, \tag{59}$$

where $\rho_L=(2\varepsilon_n)^{-1}(-\xi+\sqrt{\xi^2+4\varepsilon_n l(l+1)})$ and $\rho_R=(2\varepsilon_n)^{-1}(-\xi-\sqrt{\xi^2+4\varepsilon_n l(l+1)})$ are the turning points and $(n+1)$ is the number of nodes of $\phi(\rho)$. The integral of the momentum and the second integral in the right hand side of (59) are calculated to be

$$\int_{\rho_L}^{\rho_R}k(\rho)d\rho=\sqrt{-\varepsilon_n}\int_{\rho_L}^{\rho_R}\frac{d\rho}{\rho}\sqrt{(\rho_R-\rho)(\rho-\rho_L)}=\pi\left[\frac{\xi}{2\sqrt{-\varepsilon_n}}-\sqrt{l(l+1)}\right], \tag{60}$$

$$\int_{\rho_{L0}}^{\rho_{R0}} \rho k_0(\rho) d\rho = \sqrt{-\varepsilon_0} \int_{\rho_{L0}}^{\rho_{R0}} d\rho \sqrt{(\rho_{R0}-\rho)(\rho-\rho_{L0})} = \frac{\pi}{\xi}(l+1)^2, \tag{61}$$

we have used in the calculus the formulas given in [14].

The exact rule (59) gives the equation satisfied by ε_n

$$\varepsilon_n = -\frac{\xi^2}{4(n+l+1)^2}, \quad n = 0, 1, 2, \ldots. \tag{62}$$

In our case we have $\varepsilon_n = -1/4$, taking now into account the relations $l = -1/2 + \sqrt{(J+1/2)^2 - \gamma^2}$ and $\gamma = \alpha Z$, (62) becomes

$$\xi = n' + \sqrt{\left(J + \frac{1}{2}\right)^2 - (\alpha Z)^2} - \frac{1}{2}, \quad n' = 1, 2, 3 \ldots. \tag{63}$$

Inserting now the value of ξ given by (48) in (63) and defining the principal quantum number as $N = n' + J$, we obtain the relativistic energy spectrum for the DKP Coulomb potential

$$E_{NJ} = mc^2[1 + (\alpha Z)^2(N - J - 1/2 + \sqrt{(J+1/2)^2 - (\alpha Z)^2})^{-2}]^{-1/2}. \tag{64}$$

The result is consistent with those of [5] and [13].

In order to obtain the relativistic radial wavefunction for the DKP Coulomb problem, we put

$$F(\rho) = \rho^{l+1} e^{-\rho/2} L(\rho), \tag{65}$$

and substituting (62) with $\varepsilon_n = -1/4$ and (65) in (53), we obtain the equation

$$\rho \frac{d^2}{d\rho^2} L(\rho) + (2(l+1) - \rho) \frac{d}{d\rho} L(\rho) + n L(\rho) = 0, \tag{66}$$

which is similar to (45). Then the radial wavefunction $F(\rho)$ can be expressed as

$$F(\rho) = \lambda'_{\text{norm}} \rho^{l+1} e^{-\rho/2} L_n^{2l+1}(\rho), \tag{67}$$

where λ'_{norm} is a normalization constant. The other radial wavefunctions $H_{\pm 1}(r)$ can be obtained using (49) and (50).

5 Conclusion

The exact quantization rule as it's shown in this paper, is an alternative method to obtain the energy eigenvalues of the relativistic DKP oscillator and Coulomb problems. This practical method may be extended to others non-relativistic or relativistic interactions problems.

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